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Algebra 1

Spring Semester 2017

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Midterm

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could loose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Problem 1 :

Let G be a group and $Aut(G)$ be the automorphisms of G. We define the map

$$\begin{array}{rccc} \phi: & G & \to & Aut(G) \\ & a & \mapsto & \alpha_a \end{array}$$

where α_a is the map

$$\begin{array}{rccc} \alpha_a: & G & \to & G \\ & g & \mapsto & aga^{-1} \end{array}$$

1.(a) Give the definition of $Aut(G)$.

(b) Which group law makes $Aut(G)$ into a Group ? What is the identity element for this group ?

2. Let $a \in G$.

(a) Prove that α_a is a homomorphism.

(b) Prove that α_a is an automorphism.

3.(a) Give the definition of the center $Z(G)$ of G.

(b) Prove that $Z(G)$ is a subgroup of G.

(c) Prove that $Z(G)$ is a normal subgroup of G.

(d) Compute $Z(G)$, when G is commutative.

4.(a) Give the definition of $In(G)$, the inner automorphisms of G.

(b) Prove that $In(G)$ is a subgroup of $Aut(G)$.

(c) Prove that $In(G)$ is a normal subgroup of $Aut(G)$.

(d) Compute $In(G)$, when G is commutative.

5.(a) Prove that ϕ is an homomorphism.

(b) Compute the Kernel of ϕ and identify it with a structure seen in class.

(c) Compute the Range of ϕ and identify it with a structure seen in class.

(d) Deduce that we have the following isomorphism

$$\frac{G}{Z(G)} \simeq In(G)$$

6.(a) Compute $Z(S_3)$, where S_3 is the group of permutation between 3 objects.

(b) Deduce that

$$S_3 \simeq In(S_3)$$

(c) Is $H = \{Id, (1,2), (1,3), (1,3,2)\}$ a subgroup of S_3 ? Justify.

(d) Is $N = \{Id, (2,3)\}$ a normal subgroup of S_3 ? Justify.

(e) Is G/N a group ?

(f) Is S_3 commutative ? Justify.

(g) Compute $< (1,2), (1,3) >$ the subgroup of S_3 generated by $(1,2)$ and $(1,3)$.

(h) Compute $< (1,2,3) >$ the subgroup of S_3 generated by $(1,2,3)$. To which well known group is it isomorphic to ?

Problem 2 :

1.(a) Compute U_{12}. Justify.

(b) What is the cardinality of U_{12} ?

(c) Give the order of $[5]$ in U_{12}. Justify.

(d) Is U_{12} cyclic ? Justify.

(e) Is there an element of order 3 in U_{12} ? Justify.

2.(a) Give the order of $[2]$ is the group $\mathbb{Z}/12\mathbb{Z}$. Is it a generator for this group ? Justify.

(b) Give the list of all the generators of $\mathbb{Z}/12\mathbb{Z}$. Justify.

(c) Given an element of order 6 in $\mathbb{Z}/12\mathbb{Z}$. Justify.

(d) Given a subgroup of order 6 in $\mathbb{Z}/12\mathbb{Z}$. Justify.

(e) How many subgroup $\mathbb{Z}/12\mathbb{Z}$ has ? Justify.

(f) To which group $Aut(\mathbb{Z}/12\mathbb{Z})$ is isomorphic to ? Justify.